

# Holography: 2-D or not 2-D?

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## Abstract

As was recently pointed out by Cadoni, a certain class of two-dimensional gravitational theories will exhibit (black hole) thermodynamic behavior that is reminiscent of a free field theory. In the current letter, a direct correspondence is established between these two-dimensional models and the strongly curved regime of (arbitrary-dimensional) anti-de Sitter gravity. On this basis, we go on to speculatively argue that two-dimensional gravity may ultimately be utilized for identifying and perhaps even understanding holographic dualities.

There is a growing suspicion that the holographic principle may be a crucial element in linking together semi-classical gravity and the fundamental quantum theory. That is to say, the holographic storage of information can, perhaps, be viewed as a semi-classical manifestation of some deep, fundamental principle that has its origins in the (yet-to-be-understood) quantum nature of spacetime. (For a review on the holographic principle, see [1]. For a discussion on how it might connect with quantum gravity, see [2].)

The essence of this holographic paradigm is that the entropy (or, equivalently, the accessible information) in a given region of spacetime should have a precise limit which can be formulated in terms of the “area” of a suitably defined surface. [For a  $d$ -dimensional spacetime, this “area” would measure the volume of some  $(d - 2)$ -dimensional hypersurface.] In the “conventional” (*i.e.*, flat-space) quantum world, such a bound has contradictory implications; for instance, quantum field theory predicts that the entropy will vary extensively with the volume of the applicable region. Nonetheless, such expectations need not persist once gravitational interactions have been “turned on”. Indeed, the most strongly gravitating of objects — black holes — have a clear thermodynamic interpretation [3,4] which necessarily implies that  $S_{max} \propto A$  [5]; with  $S_{max}$  being the maximal amount of entropy that can be stored in a region bounded by a surface of area  $A$ . Analogous bounds can be extrapolated to other scenarios (both strongly and weakly gravitating) by way of the so-called covariant entropy bound [6]. So far, no violations of this bound are known given reasonable conditions on what constitutes physically allowable matter [1].

Although not entropy bounds *per se*, dualities between (bulk) gravitational and (boundary) field theories provide another elegant realization of the holographic principle. The most notable of these being the duality that is known to exist between an anti-de Sitter spacetime

and a conformal field theory of one dimension fewer; that is, the celebrated “AdS/CFT” correspondence [7–9]. Such a duality is naturally holographic in the following sense: If the correspondence is truly complete (*i.e.*, one to one), then the lower-dimensional field theory provides a strict means of limiting the amount of information that can be stored in the bulk spacetime. Moreover, given that the field theory typically “lives” on a boundary of the spacetime, this entropic bound can be precisely related to the area of some exterior surface.

An important distinction between the covariant entropy bound and these “holographic dualities” is that the former is believed (at least hopefully) to have universal validity whereas the latter certainly does not. That is to say, it is quite clear that a field theory dual does *not* exist for every type of gravitational theory. (Note, however, that this non-universality should not be viewed as a failure in the underlying principle. See [1] for further discussion on this point.) One might then be inclined to wonder what fundamental principle makes this “decision” or, to rephrase, what exactly determines when a duality does or does not exist? Although the definitive answer will almost certainly require a rigorous notion of quantum gravity, it still behooves us to see if progress can be made via semi-classical considerations. In this regard, an important first step might be to establish criteria for the existence/non-existence of a holographic dual.

In a recent paper [10], Cadoni considered various aspects of the holographic principle in the context of two-dimensional dilaton-gravity theories. (For other pertinent work, see [11]. For a general review of two-dimensional gravity, see [12].) An interesting observation (actually, one of many) was that, for a certain restricted class of these dilatonic models, the black hole thermodynamic behavior effectively mimics that of a free field theory. More precisely, the entropy ( $S$ ) and the energy (or black hole mass,  $M$ ) are related, for models with a power-law (dilaton) potential, in accordance with  $S \sim M^{p/(p+1)}$  (where  $p^{-1}$  is the “power”). It thus follows that, when  $p$  is a positive integer, this relation mimics the thermodynamic behavior of a field theory of dimensionality  $p + 1$ .<sup>1</sup>

The main point of the current treatment is to demonstrate, quite rigorously, that this special class of two-dimensional theories can be identified with a dimensionally reduced form of (“strongly curved”) anti-de Sitter gravity. Moreover, the dimensionality of the anti-de Sitter spacetime,  $d = n + 2$ , fixes the parameter  $p$  such that  $p = n$ . Hence, one obtains an effective field theory of dimensionality  $n + 1$ , in compliance with the expectations of the AdS/CFT correspondence. Following the formal analysis, we will explain how this outcome could be significant in the context of our earlier discussion.

To begin here, let us consider the gravitational action for an anti-de Sitter spacetime of arbitrary dimensionality ( $d = n + 2 > 3$ ). That is,<sup>2</sup>

$$I^{(n+2)} = \frac{1}{16\pi l^n} \int d^{n+2}x \sqrt{-g^{(n+2)}} \left[ R^{(n+2)} + \frac{n(n+1)}{L^2} \right], \quad (1)$$

where  $l^n$  is the  $n+2$ -dimensional Newton constant and  $L$  is the anti-de Sitter curvature

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<sup>1</sup>This notion of black hole thermodynamics exhibiting an *effective* dimensionality has been previously considered in the context of critical-point behavior [13].

<sup>2</sup>The speed of light, Boltzmann’s constant and Planck’s constant will be set to unity throughout.

radius. Note that  $\Lambda = -\frac{n(n+1)}{2L^2}$  is the negative cosmological constant, and so  $L^{-2}$  measures the “strength” of the curvature.

Now, to reduce this *fundamental* action into that of an *effective* two-dimensional theory, we will utilize the following spherical *ansatz*:

$$ds_{n+2}^2 = ds_2^2(t, x) + \phi^2(t, x)d\Omega_n^2 . \quad (2)$$

With this choice, it is straightforward (albeit tedious) to show that equation (1) transforms into

$$I = \frac{\mathcal{V}_n}{16\pi l^n} \int d^2x \sqrt{-g} \phi^n \left[ R + n(n-1) \left( \frac{(\nabla\phi)^2}{\phi^2} + \frac{1}{\phi^2} \right) + \frac{n(n+1)}{L^2} \right] , \quad (3)$$

where  $\phi$ , “the dilaton”, can be identified with the radius of the symmetric  $n$ -sphere,  $\mathcal{V}_n$  is the dimensionless volume of a unit  $n$ -sphere, and all geometric quantities are now defined with respect to the resultant 1+1-manifold.

For convenience, let us (following [14,15]) redefine the dilaton field as follows:

$$\psi(x, t) \equiv \left[ \frac{\phi}{l} \right]^{\frac{n}{2}} . \quad (4)$$

The reduced action (3) can then be re-expressed in the following manner:

$$I = \frac{1}{2G} \int d^2x \sqrt{-g} \left[ D(\psi)R + \frac{1}{2} (\nabla\psi)^2 + \frac{1}{l^2} V(\psi) \right] , \quad (5)$$

where we have also defined

$$\frac{1}{2G} \equiv \frac{8(n-1)\mathcal{V}_n}{16\pi n} , \quad (6)$$

$$D(\psi) \equiv \frac{n}{8(n-1)} \psi^2 , \quad (7)$$

$$V(\psi) \equiv \frac{n^2}{8} \psi^{(2n-4)/n} + \frac{(n+1)n^2 l^2}{8(n-1)L^2} \psi^2 . \quad (8)$$

Note that  $G$  can be interpreted as the (dimensionless) gravitational coupling for the reduced theory.

Significantly, the above form of action allows us to directly implement a special field reparametrization that eliminates the kinetic term [16]. More explicitly, following the methodology of the cited paper, we first redefine the dilaton, metric and “dilaton potential” as follows:

$$\bar{\psi} = D(\psi) \equiv \frac{n}{8(n-1)} \psi^2 , \quad (9)$$

$$\bar{g}_{\mu\nu} \equiv \Omega^2(\psi) g_{\mu\nu} , \quad (10)$$

$$\overline{V}[\overline{\psi}(\psi)] \equiv \frac{V(\psi)}{\Omega^2(\psi)}, \quad (11)$$

where

$$\begin{aligned} \Omega^2(\psi) &\equiv \exp \left[ \frac{1}{2} \int \frac{d\psi}{(dD/d\psi)} \right] \\ &= \mathcal{C} \psi^{2(n-1)/n} = \mathcal{C} \left[ \frac{8(n-1)\overline{\psi}}{n} \right]^{\frac{n-1}{n}}. \end{aligned} \quad (12)$$

In the last line [which, incidentally, made use of equation (9)],  $\mathcal{C}$  represents a seemingly arbitrary constant of integration. Nevertheless, this constant can be fixed via physical arguments (see [14]) so that  $\mathcal{C} = n^2/8(n-1)$ . For future reference, some straightforward evaluation yields

$$\overline{V}(\overline{\psi}) = (n-1) \left[ \frac{8(n-1)\overline{\psi}}{n} \right]^{-\frac{1}{n}} + (n+1) \frac{l^2}{L^2} \left[ \frac{8(n-1)\overline{\psi}}{n} \right]^{+\frac{1}{n}}. \quad (13)$$

In terms of the above parametrizations, the reduced action (5) now takes the simplified form

$$I = \frac{1}{2G} \int d^2x \sqrt{-\overline{g}} \left[ \overline{\psi} R(\overline{g}) + \frac{1}{l^2} \overline{V}(\overline{\psi}) \right]; \quad (14)$$

and we will subsequently drop the cumbersome “overline” notation. It should be emphasized that this form of the action is perfectly general. That is, a generic two-dimensional dilaton-gravity theory can always be translated into the form of equation (14) (given some modest constraints on the initial action; see [16] for details).

With the gauge choice  $\psi = x/l \geq 0$ , the general solution of this effective action reveals a static, Schwarzschild-like metric

$$ds^2 = -F(x)dt^2 + F^{-1}(x)dx^2, \quad (15)$$

where

$$F(x) \equiv J(x) - 2lGM, \quad (16)$$

with  $M \geq 0$  representing the conserved mass [17] and

$$J[\psi(x)] \equiv \int^{\psi=x/l} V(\tilde{\psi}) d\tilde{\psi}. \quad (17)$$

(Note that the integration constant has already been incorporated into the observable  $M$ .) Integrating equation (13), we specifically obtain

$$J(\psi) = n \left[ \frac{8(n-1)}{n} \right]^{-\frac{1}{n}} \psi^{(n-1)/n} + n \frac{l^2}{L^2} \left[ \frac{8(n-1)}{n} \right]^{\frac{1}{n}} \psi^{(n+1)/n}. \quad (18)$$

In a general sense, any form of  $J(\psi)$  — or alternatively  $V(\psi)$  — which admits black hole solutions (see [18] for the relevant criteria) leads to a well-defined notion of the associated thermodynamics (even though there is no strict analogy to the horizon area in a two-dimensional spacetime). In particular, the temperature and entropy are respectively identifiable as [19]

$$T = \frac{1}{4\pi l} V(\psi_h) , \quad (19)$$

$$S = \frac{2\pi}{G} \psi_h . \quad (20)$$

Here,  $\psi = \psi_h$  locates the event horizon (assuming one exists); that is,  $F[\psi(x)] = 0$  when  $\psi = \psi_h$  or  $J(\psi_h) = 2lGM$ .

Now, returning to our reduced anti-de Sitter model, it is clear that an event horizon does indeed exist; inasmuch as  $J(\psi)$  is strictly non-negative for any admissible  $\psi$ . It becomes a difficult problem to solve analytically for  $\psi_h$  under general circumstances. Nevertheless, the situation noticeably improves if one considers a regime of “small  $L$ ”; essentially,  $L^2 \ll \phi^2$ , where we recall that  $\phi$  is the radius of the  $n$ -sphere. [If we consider the relevant higher-dimensional solution — an anti-de Sitter–Schwarzschild black hole in static coordinates — then it is readily confirmed that this condition translates into a lower bound on the black hole mass:  $M_{AdS-S} \gg L^{n-1}l^{-n}$  (as also discussed in [10].) Since our (implied) semi-classical analysis will, in all likelihood, breakdown for sub-Planckian scales or when  $L^2 < l^2$ , this bound really indicates the regime of a macroscopically large black hole (in Planck units). Further note that, from the fundamental or higher-dimensional perspective, this regime translates into one of high temperature<sup>3</sup> or, alternatively, one in which the (negative) curvature of the spacetime can not, even locally, be discounted.] In this case, one can verify — with the help of equations (4) and (9) — that the right-most term dominates equation (18), and so

$$J(\psi) \approx n \frac{l^2}{L^2} \left[ \frac{8(n-1)}{n} \right]^{\frac{1}{n}} \psi^{(n+1)/n} ; \quad (21)$$

moreover,

$$\psi_h \approx \left[ \frac{n}{8(n-1)} \right]^{\frac{1}{n+1}} \left[ \frac{2GL^2M}{nl} \right]^{\frac{n}{n+1}} ; \quad (22)$$

which is to say,

$$S = \frac{2\pi}{G} \psi_h \sim M^{n/(n+1)} . \quad (23)$$

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<sup>3</sup>This can readily be observed with an inspection of the Hawking temperature for an anti-de Sitter–Schwarzschild black hole. See, for instance, [20].

The interesting point about the above relation is that it effectively mimics an  $n + 1$ -dimensional field theory with entropy  $S$  and energy  $E = M$ . Given that our starting point is  $n + 2$ -dimensional anti-de Sitter space, this outcome can best be interpreted as a manifestation of the AdS/CFT correspondence (although the *effective* field theory described above is certainly not the one usually referred to when this duality is discussed). Nevertheless, the relation  $S \sim M^{n/(n+1)}$  could have similarly (and rather more easily) been deduced by looking directly at the small- $L$  limit of the (fundamental) anti-de Sitter–Schwarzschild metric [10]. Hence, it is reasonable to ask if anything has really been gained by translating the theory into a two-dimensional context.

To address this issue of relevance, let us first elaborate on a point made by Cadoni [10]. Given a dimensionally reduced action with the generic form of equation (14) [ignoring the overlines], one finds that, for a power-law potential or <sup>4</sup>

$$V(\psi) \sim \psi^{1/p} \quad \text{with} \quad p^{-1} > -1 , \quad (24)$$

the entropy-mass relationship becomes

$$S \sim M^{p/(p+1)} . \quad (25)$$

Hence, as pointed out in [10], one always obtains field-theory-like thermodynamics whenever  $p$  is equal to a positive integer; with  $p+1$  being the dimensionality of this *effective* field theory. The model considered in the current letter can thus be viewed as a concrete realization of this previously observed phenomena. More specifically, we have found  $V(\psi) \sim \psi^{1/n}$  (in the regime of interest), and the effective field theory is consequently of dimensionality  $n + 1$ .

In view of the above, we propose that our analysis provides a clear illustration of how dimensional-reduction techniques may be employed for identifying so-called holographic dualities (as discussed earlier in the letter). To elaborate, let us consider a hypothetical (higher-dimensional) gravity theory for which an entropy-mass relation can not be so readily deduced from the fundamental metric. The above techniques could be purposefully utilized to cast the theory into the reduced form of equation (14). It then becomes a matter of inspecting the potential for some viable regime (or regimes) in which  $V(\psi) \sim \psi^{1/p}$  such that  $p$  is a positive integer. If such a regime does indeed exist, it would seem quite probable that some sort of holographic duality will be in effect. That is to say, two-dimensional dilatonic gravity may yet provide us with an intriguing means for clarifying and, much more speculatively, understanding holographic dualities.

Finally, it is interesting to note that, by way of the above reasoning, asymptotically flat space can *not* have any such field theory dual. (Also discussed in [10].) To see this most definitively, consider the form of our potential [*cf.* equation (13)], but now in the asymptotically flat limit of  $L \rightarrow \infty$ . In this case,  $V(\psi) \sim \psi^{-1/n}$  or  $p = -n < 0$ , and it becomes clear that a field-theory description is no longer possible. Nonetheless, the inclusion of charge or more exotic “hair” may possibly paint a different picture (at least for some suitable choices of regime); a topic which is currently under investigation.

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<sup>4</sup>Note that our parameter  $p$  is the *inverse* of the parameter  $h$  used in [10].

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